EXAM ALGEBRAIC STRUCTURES,

June 21st, 2019, 9.00pm–12.00pm, MartiniPlaza, L. Springerlaan 2.

Please provide complete arguments for each of your answers. The exam consists of 3 *questions each subdivided into* 4 *parts. You can score up to* 3 *points for each part, and you obtain* 4 *points for free.*

In this way you will score in total between 4 *and* 40 *points.*

(1) In this exercise we denote the ring $\mathbb{Z}[t]/(t^3)$ by R. Elements of R we write as $f(t) \bmod (t^3)$, for some $f(t) \in \mathbb{Z}[t]$.
  (a) Show that $t+1 \bmod (t^3)$ is a unit in R and find its inverse.
  (b) Does, apart from $1 \bmod (t^3)$ and $0 \bmod (t^3)$, the ring R contain any idempotent (i.e., an element $\gamma \in R$ with $\gamma^2 = \gamma$)?
  (c) Show that no unitary rings R_1 and R_2 exist in which $0 \neq 1$, such that $R \cong R_1 \times R_2$.
  (d) For $a, b, c \in \mathbb{Z}$, show that $a + bt + ct^2 \bmod (t^3)$ is a unit in R, if and only if $a = \pm 1$.

(2) Consider the ring $R = \mathbb{Q}[x, y]$.
  (a) Show that if $P \subset R$ is a prime ideal, then $P \cap \mathbb{Q}[x]$ is a principal ideal in $\mathbb{Q}[x]$ that is either generated by 0 or by an irreducible element of $\mathbb{Q}[x]$.
  (b) Show that $\mathbb{Q}[x, y] \cdot (x - y^2)$ is a prime ideal in R.
  (c) Show that $x^3 + y^3 + 1 \in R$ is irreducible.
  (d) Prove that the ideal in R generated by the two polynomials $x - y^2$ and $x^3 + y^3 + 1$ is a maximal ideal in R.

(3) In this final exercise, R denotes the field $\mathbb{F}_2[t]/(t^4 + t + 1)$.
  (a) Show that indeed R is a field.
  (b) Find the minimal polynomial of $t^2 + t \bmod (t^4 + t + 1)$ over the prime field of R.
  (c) Show that $f(t) \bmod (t^4 + t + 1) \mapsto f(t+1) \bmod (t^4 + t + 1)$ is a well-defined automorphism of the field R.
  (d) What are the possible orders of elements in the group of units $R^{\times}$?